Assignment 5:

Instructions

Complete the following problem set showing your work. Problems may be worked out "by hand" or in "python" or with the assistance of other analytical software (e.g., Mathematica, MatLab). You may use chatGPT to assist in coding.

Solutions must be type written (e.g., in Jupyter, markdown, or latex). Upload PDF solution by question to crowdmark (link will be emailed to you) by **11:59pm** on the **Sunday** of the corresponding week (see syllabus). If you have issues with Crowdmark submission please email solutions to Rebekah Hall (rah11@sfu.ca).

All problems are equally weighted within an assignment. Students in 468 may or may not choose to attempt the challenge question for a bonus pts. Students in 795 are required to complete the challenge question.

Problem Set

1. Lotka-Voterra Model

Consider the Lotka-Volterra Model of describing the number of prey $X(t)$ (e.g., hares) and predators $Y(t)$ (e.g., lynx) as described by the system of coupled differential equations with the harvesting of the prey and predators by the Hudson Bay Company. Here we consider a semi-sustainable model of harvest where individuals are removed in proportion to their density to the power a, N^a where $a > 1$. This ensures that the rate of harvest drops quickly as the number of individuals declines

$$\frac{dX}{dt} = \underbrace{\alpha X}_{\text{Prey Birth}} - \underbrace{\beta XY}_{\text{Prey death}} - \underbrace{\mu_X X^{1.5}}_{\text{harvest}}$$
$$\frac{dY}{dt} = \underbrace{\delta XY}_{\text{Preditor birth}} - \underbrace{\gamma Y}_{\text{Preditor Birth}} - \underbrace{\mu_Y Y^{1.5}}_{\text{harvest}}$$

Part A. Numerically integrate the ODEs above for $\alpha = 1$, $\beta = 0.03$, $\delta = 0.01$ and $\gamma = 0.2$ and $\mu_X = \mu = Y = 0.01$ assuming we start with $X(0) = 500$ and $Y(0) = 200$.

Part B. Propose an analogous continuous time stochastic process for this model. Describe the 2D state space of this model.

Part C. Simulate the dynamics of the stochastic model you proposed and compare the simulated trajectories to your answer in part 1.

Part D: Derive a system of master equations for the probability of having n prey and m predators at time t in the absence of harvesting $\mu_X = \mu_Y = 0$. You do NOT need to solve them numerically.

2. SIS Model

Consider the model given in example 5.13: a stochastic SIS model where transmissions occur at a mass-action rate of $\frac{\beta}{\kappa} * S * I$ with $\beta = 0.5$, hosts recover (becoming susceptible again) at a rate $\gamma = 0.1$ and the total population size is $\kappa = 100$.
*Our goal here is to model the mean and variance in the number of infections in this model using an ensemble moment approximation.

Part A. What are the ODEs that describe the dynamics of $\langle I \rangle$ and $\langle I^2 \rangle$? How do these equations depend on $\langle I^3 \rangle$ and why?

Part B. Implement moment closure assuming that the skew in the number of infections is small.

Part C. Numerically solve the dynamics for the mean and variance assuming $\beta = 0.5, \gamma = 0.1, \kappa = 100$ and $I(0) = 20$ and no variance for the first 10 units of time.

Part D. Challenge 795: Using your numerical solution in 3 comment on whether your approximation is valid for the time period considered.

Part E. Challenge 795: Compare your solution to the EMA to the result you obtain from a simulation approach.

3. The Yule Process

In the lecture we derived the master equation for the size of a clade in the Yule model:

$$\frac{dP_n(t)}{dt} = -\lambda n P_n + \lambda(n-1)P_{n-1} \quad P_n(0) = \begin{cases} 0 & n \neq 1 \\ 1 & n = 1 \end{cases}$$

with the solution:

$$P(n,t) = e^{-n\lambda t}\left(e^{\lambda t} - 1\right)^{(n-1)}$$

Part A. Challenge for 795: Show that this is the case

Part B: Plot the distribution of clade sizes at $\lambda = 1$ at $T = 1.5$ for $n = 1 \ldots 20$

Part C: Technically the number of lineages at time T could be infinite, obviously we can make a plot in part 2 with an infinite domain. How close is the approximation of the probability distribution you showed in part 2 to the truth? In other words how much of the total probability are you missing by showing a plot on a finite domain?

Part D: Simulate 50 trees in the Yule model for these parameters. Do your simulations match the solutions from the master equation?

4. The Birth-Death Process

Part A: Propose a system of Master Equations describing the size of a clade in the birth death model.

Part B: Numerically solve these equations assuming $\lambda = 1.5$ and $\mu = 0.5$ and for $t < T = 0.5$.

Part C: How does clade size in this model compare to the Yule model above? Are you surprised they are the same/different?