The Coalescent as a modified Poisson Process

continuous time
discrete state

Review of Const Rate Dist

A) Exponential Dist

→ waiting time 1st event

$$Pr(T=t) = \lambda e^{-\lambda t}$$

$$E[T] = \frac{1}{\lambda}$$

$$Var[T] = \frac{1}{\lambda^2}$$

B) Poisson Distribution

→ # of events in $T=1$ time

$$Pr(N=k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$E[N] = \lambda \qquad Var[N] = \lambda$$

C) Erlang Distribution

→ waiting time to event k

$$Pr(T=t \mid k) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!}$$

$$E[T] = \frac{k}{\lambda} \qquad Var[T] = \frac{k}{\lambda^2}$$

Deriving the Erlang Dist for $k=2$

$$Pr(T \mid k=2) = \int_0^T \lambda e^{-\lambda t_1} \lambda e^{-\lambda (T-t_1)} dt_1$$

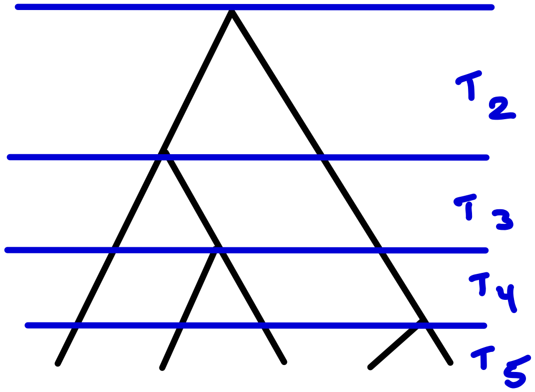
$$= \lambda^2 \int_0^T e^{-\lambda T} dt_1 = \lambda^2 e^{-\lambda T} \int_0^T 1 \, dt_1$$

$$= \lambda^2 e^{-\lambda T} \cdot T$$

Convolution

Summary Statistics Overview

Coalescent Times (Exponential Analog)



$$Pr(T_i = \tau_i) = \binom{i}{2} e^{-\binom{i}{2}\tau_i}$$

$$E[\tau_i] = \frac{1}{\binom{i}{2}}$$

$$Var[\tau_i] = \frac{1}{\binom{i}{2}^2}$$

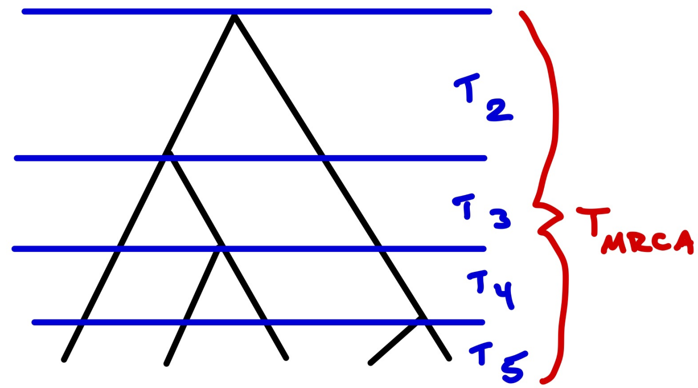
Convolution

$$Pr(t_1 + t_2 = \tau) =$$

$$\int_0^{\tau} \lambda_1 e^{-\lambda_1 t} \lambda_2 e^{-\lambda_2(\tau - t)} \, dt$$

t_1 $\quad$ $t_2 = \tau - t_1$

$$T_{MRCA} = \sum_{i=2}^{n} T_i$$

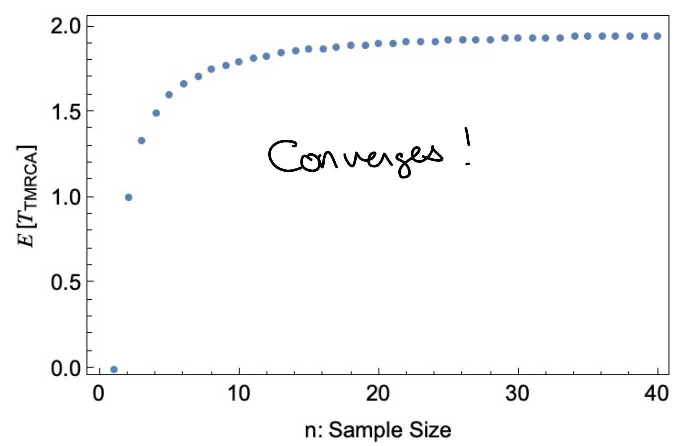


$$Pr(T_{MRCA} = t) = \sum_{i=2}^{n} \binom{i}{2} e^{-\binom{i}{2}t} \prod_{\substack{j=2 \\ j \neq i}}^{n} \frac{\binom{j}{2}}{\binom{j}{2} - \binom{i}{2}}$$

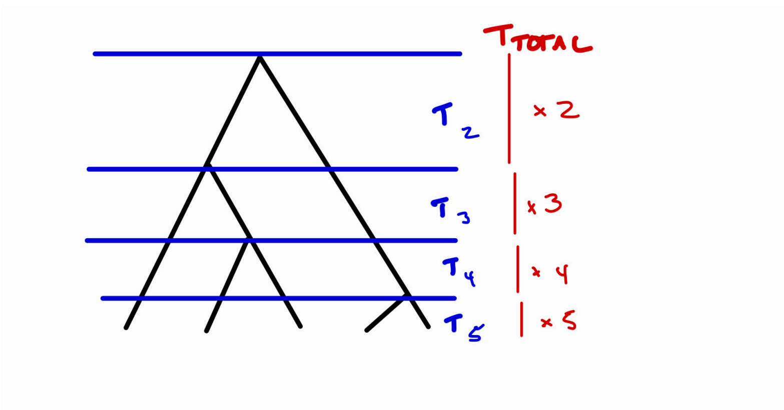
* We can calculate these quantities by taking the convolution of exp. dist.

$$E[T_{MRCA}] = 2\left(1 - \frac{1}{n}\right)$$

* We can calculate this by switching the order of the sum and the expectation



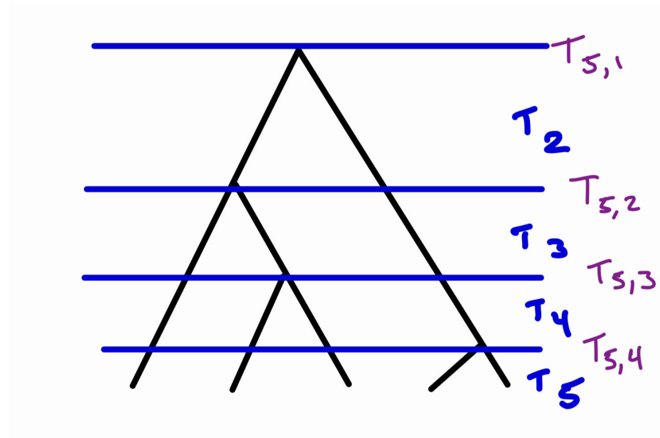
$$T_{TOTAL} = \sum_{i=2}^{n} i\,T_i$$



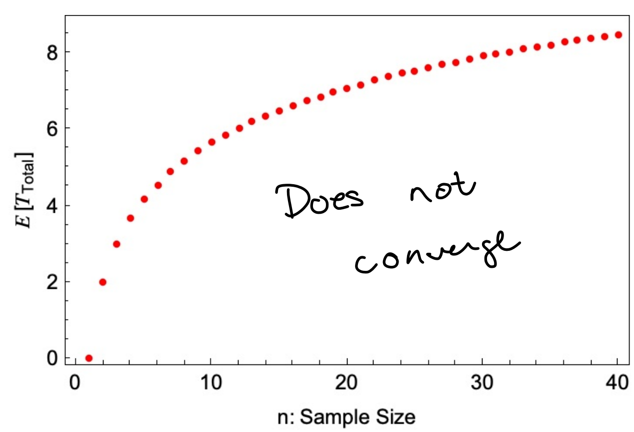
Analogous to Erlang

Time to k ancestors $\quad T_{n,k} = \sum_{i=n-k}^{n} T_i$

(Time to the n-k coal event)



$$E[T_{TOTAL}] = \sum_{i=2}^{n} \frac{2}{i-1}$$



Poisson Analog

Probability of k Ancestors at time τ

$$\Pr\left(A_n(\tau)=k\right) = \frac{1}{\binom{k}{2}} \sum_{i=k}^{n} \binom{i}{2} e^{-\binom{i}{2}\tau} \prod_{\substack{j=k \\ j\neq i}} \frac{\binom{j}{2}}{\binom{j}{2}-\binom{i}{2}}$$